Vertex Operator Algebra Analogue of Embedding D_8 into E_8

Yan-Jun Chu, Zhu-Jun Zheng*

Abstract: Let $L_{D_8}(1,0)$ and $L_{E_8}(1,0)$ be the simple vertex operator algebras associated to untwisted affine Lie algebra $\widehat{\mathbf{g}}_{D_8}$ and $\widehat{\mathbf{g}}_{E_8}$ with level 1 respectively. In the 1980s by I. Frenkel, Lepowsky and Meurman as one of the many important preliminary steps toward their construction of the moonshine module vertex operator algebra, they use roots lattice showing that $L_{D_8}(1,0)$ can embed into $L_{E_8}(1,0)$ as a vertex operator subalgebra([5, 6, 8]). Their construct is a base of vertex operator theory. But the embedding they gave using the fact $L_{\mathbf{g}}(1,0)$ is isomorphic to its root lattice vertex operator algebra V_L. In this paper, we give an explicitly construction of the embedding and show that as an $L_{D_8}(1,0)$-module, $L_{E_8}(1,0)$ is isomorphic to the extension of $L_{D_8}(1,0)$ by its simple module $L_{D_8}(1,\overline{\omega}_8)$. It may be convenient to be used for conformal field theory.
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1 Introduction

Affine Lie algebras plays a critical role in the construction of conformal field theories. While conformal embeddings of affine Lie algebras preserves conformal invariance([1]). In string theory, conformal invariance become very important. A key of Frenkel-Kac Compactification of bosonic strings is that it respects conformal invariance. And conformal embeddings of affine Lie algebras just guarantee the preservation of conformal invariance, which work in the construction of bosonic strings with nonsimply-laced gauge groups([2, 3, 4]).

In the 1980s, I. Frenkel, Lepowsky and Meurman gave out the constuction of the moonshine module vertex operator algebra. As an important

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preparation, they constructed untwisted vertex operators and vertex operator representations using the extensions of lattices. At first, given an nondegenerate positive definite even lattice L, they constructed a class of Lie algebra $\mathbf{g}$ by the extension of L, and got the untwisted affine Lie algebra $\tilde{\mathbf{g}}$. Next, they constructed untwisted vertex operators and vertex operator representation $V_L = S(\widehat{\eta_{\mathbb{Z}}^-}) \otimes \mathbb{C}\{L\}$ of $\tilde{\mathbf{g}}$, where $\eta = L \otimes_{\mathbb{Z}} \mathbb{C}$. Moreover, since L is a positive definite even lattice, the vertex operator representations V_L has the structure of vertex operator algebra, and the conformal vector is given by $\omega = \frac{1}{2}\sum_{i=1}^{d} h_i(-1)^2$, where $h_1, h_2, \cdots, h_d$ is an orthonormal basis of $L \otimes_{\mathbb{Z}} \mathbb{C}$([8]). By the process of construction of the representation V_L or I.Frenkel-Kac-Segal Construction([5, 6]), if L is the root lattice of a simple Lie algebra $\tilde{\mathbf{g}}$ of types A_n, D_n, E_n, the lattice vertex operator algebra V_L is isomorphic to the simple affine vertex operator algebra $L_{\tilde{\mathbf{g}}}(1,0)$ with level 1. At the same times, they pointed out that if a positive definite even lattice can embed into another positive definite even lattice with the same ranks, there are the corresponding embedding relations between these two vertex operator algebras with the same conformal vectors. Let Q_{D_8}, Q_{E_8} be the roots lattices of simple Lie algebras $\mathbf{g}_{D_8}, \mathbf{g}_{E_8}$, respectively. Since Q_{D_8} can embed into Q_{E_8} with the same rank 8, hence the simple affine vertex operator algebra $L_{D_8}(1,0)$ can embed into the affine vertex operator algebra $L_{E_8}(1,0)$ as a subalgebra with the same conformal vectors.

Considering the non-trivality of isomorphisms between V_L and $L_{\mathbf{g}}(1,0)$, and the importance of $L_{D_8}(1,0)$ and $L_{E_8}(1,0)$ in conformal field theory, we give an explicit construction of the embedding $L_{D_8}(1,0)$ into $L_{E_8}(1,0)$ as a vertex operator subalgebra without using the isomorphic relations to the lattice vertex operator algebras.

Another main motivation of our study is that O. Perše's study of vertex operator algebra analogue of embedding of B_4 into F_4([12]). Let $L_{B_4}(-\frac{5}{2},0)$ and $L_{F_4}(-\frac{5}{2},0)$ be the simple vertex operator algebra associated to simple Lie algebras of type B_4 and F_4, respectively, with the conformal vectors obtained by Segal-Sugawara construction. In the case of admissible level $k = -\frac{5}{2}$, the maximal proper submodules of $N_{B_4}(-\frac{5}{2},0)$ is generated by one singular vector([13]). Using the equality related to the singular vector, O. Perše showed that the conformal vectors of $L_{B_4}(-\frac{5}{2},0)$ and $L_{F_4}(-\frac{5}{2},0)$ are same by means of straightforward calculation, then proved $L_{B_4}(-\frac{5}{2},0)$ can embed into $L_{F_4}(-\frac{5}{2},0)$ as a vertex operator subalgebra(cf. [12]).

It's sound that the idea of O. Perše's([12]) can be transplanted here to study the relations of $L_{E_8}(1,0)$ and $L_{D_8}(1,0)$. But we find that because of complexity of the structure of Lie algebra E_8's root system, to find the analogue equality related to the singular vector is almost impossible. For simple Lie algebras $\mathbf{g}_{E_8}$ and $\mathbf{g}_{D_8}$ of type E_8 and D_8, to obtain a similar result, we have to find a different method.

Next we summarize our main construction. It's known that $\mathbf{g}_{D_8}$ is a Lie

subalgebra of $\mathbf{g}_{E_8}$, and as a $\mathbf{g}_{D_8}$-module, the decomposition of $\mathbf{g}_{E_8}$ is

$$\mathbf{g}_{E_8} = \mathbf{g}_{D_8} \oplus V_{D_8}(\overline{\omega}_8), \tag{1.1}$$

where $V_{D_8}(\overline{\omega}_8)$ is the irreducible highest weight $\mathbf{g}_{D_8}$-module whose highest weight is the fundamental weight $\overline{\omega}_8$ for $\mathbf{g}_{D_8}$. For $k \in \mathbb{C}$, denote by $L_{D_8}(k,0)$ and $L_{E_8}(k,0)$ to be the simple vertex operator algebras associated to $\mathbf{g}_{D_8}$ and $\mathbf{g}_{E_8}$ with level k and conformal vectors obtained from Segal-Sugawara construction, respectively. If $L_{D_8}(k,0)$ is a vertex subalgebra of $L_{E_8}(k,0)$ with the same conformal vector, the equality of conformal vectors implies the following equality of corresponding central charges

$$\frac{k dim \widehat{\mathbf{g}}_{E_8}}{h^{\vee}_{E_8} + k} = \frac{k dim \widehat{\mathbf{g}}_{D_8}}{h^{\vee}_{D_8} + k}. \tag{1.2}$$

The equation has only solution $k = 1$. It implies the only possibility of embedding of vertex operator algebra $L_{D_8}(k,0)$ into $L_{E_8}(k,0)$ has only one case $k = 1$.

Let $\omega_{E_8}, \omega_{D_8}$ be respectively conformal vectors of $L_{E_8}(1,0)$, $L_{D_8}(1,0)$ obtained by Segal-Sugawara construction. For the case of $k = 1$, we note that $L_{E_8}(1,0)$ is a weak $L_{D_8}(1,0)$-module, and using the regularity of vertex operator algebra $L_{D_8}(1,0)$, $L_{E_8}(1,0)$ is a sum direct of finitely many simple $L_{D_8}(1,0)$-modules. Associating to the properties of vertex algebra modules, we show that $L_{D_8}(0) = L_{E_8}(0)$ on $L_{E_8}(1,0)$, so the conformal vector ω_{D_8} of $L_{D_8}(1,0)$ is also a conformal vector of $L_{E_8}(1,0)$. By some tedious and complicated calculations, we show that for $n \in \mathbb{Z}$, $L_{D_8}(n) = L_{E_8}(n)$ as operators acting on $L_{D_8}(1,0)$-module $L_{E_8}(1,0)$. Since $L_{E_8}(1,0)$ is a simple vertex operator algebra, it can be shown that $\omega_{E_8} = \omega_{D_8}$ as conformal vectors of $L_{E_8}(1,0)$, i.e. $L_{D_8}(1,0)$ can embed into $L_{E_8}(1,0)$ as a vertex operator subalgebra. Moreover, we give the direct sum decomposition of $L_{E_8}(1,0)$ as a $L_{D_8}(1,0)$-module. In addition, O. Perše also studied vertex operator algebras associated to affine Lie algebras $\widehat{A}_l, \widehat{B}_l$ and $\widehat{F}_4$ with admissible half-integer levels and the corresponding embedding relations([12, 13, 14]).

2 Vertex operator algebras associated to affine Lie algebras

2.1 Vertex operator algebras and modules

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. The triple $(V, Y, \mathbf{1})$ carries the structure of a vertex algebra(cf. [7, 8, 9, 11]), and ω is a conformal vector of vertex algebra $(V, Y, \mathbf{1})$.

A vertex subalgebra of vertex algebra V is a subspace U of V such that $\mathbf{1} \in U$ and $Y(a,t)U \subset U[[t, t^{-1}]]$ for any $a \in U$. Assume that $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra and $(U, Y, \mathbf{1}, \omega')$ is a vertex subalgebra of V, that

has a structure of vertex operator algebra, then it is said that U is vertex operator subalgebra of V if $\omega = \omega'$.

An ideal of a vertex operator algebra V is a subspace I of V such that $Y(u,t)v \in I[[t,t^{-1}]]$, for any $u \in I, v \in V$. V is said to be simple if V is the only nonzero ideal. Given an ideal I in V, such that $\mathbf{1} \notin I, \omega \notin I$, then the quotient V/I admits a natural vertex operator algebra structure(cf. [12]).

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. A weak V-module M(cf. [12, 16, 17]) is a vector space equipped with a linear map

$$\begin{aligned} Y_M :& V \longrightarrow (\mathrm{End}M)[[t,t^{-1}]] \\ & v \longmapsto Y_M(v,t) = \sum_{n\in\mathbb{Z}} v_n t^{-n-1} (v_n \in \mathrm{End}M), \forall v \in V \end{aligned}$$

satisfying the following conditions: for $u, v \in V, w \in M$,

$$v_n w = 0, \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large;} \tag{2.1}$$

$$Y_M(\mathbf{1}, t) = \mathrm{Id}_M; \tag{2.2}$$

$$\begin{aligned} t_0^{-1}\delta\left(\frac{t_1-t_2}{t_0}\right) Y_M(u,t_1)Y_M(v,t_2) - t_0^{-1}\delta\left(\frac{t_2-t_1}{-t_0}\right) Y_M(v,t_2)Y_M(u,t_1) \\ = t_2^{-1}\delta\left(\frac{t_1-t_0}{t_2}\right) Y_M(Y(u,t_0)v,t_2); \end{aligned} \tag{2.3}$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}(\mathrm{rank}V), \tag{2.4}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1}, \quad i.e.\ Y_M(\omega, t) = \sum_{n\in\mathbb{Z}} L(n)t^{-n-2}.$$

$$\frac{d}{dt}Y_M(v,t) = Y_M(L(-1)v,t). \tag{2.5}$$

We denote this module by (M, Y_M).

Lemma 2.1. *([16]) Relations* (2.4) *and* (2.5) *in the definition of weak module are consequences of* (2.1)—(2.3).

Definition 2.2. ([16]) An admissible V-module is a weak V-module M which carries a $\mathbb{N}$-grading structure $M = \bigoplus_{n\in\mathbb{N}} M(n)$ satisfying the condition: if $r, m \in \mathbb{Z}, n \in \mathbb{N}$ and $a \in V_r$, then there are

$$a_m \cdot M(n) \subseteq M(r+n-m-1). \tag{2.6}$$

Definition 2.3. An (ordinary) V-module is a weak V-module M and $L(0)$ acts semi-simply on M with the decomposition into $L(0)$-eigenspaces $M = \bigoplus_{\lambda\in\mathbb{C}} M(\lambda)$ such that for any $\lambda \in \mathbb{C}, \mathrm{dim}M(\lambda) < +\infty$ and $M(\lambda+n) = 0$ for $n \in \mathbb{Z}$ sufficiently small.

Definition 2.4. An admissible V-module M is simple if it has only 0 and M itself as its $\mathbb{N}$-grading submodules.

Definition 2.5. ([9, 16, 17]) V is called rational if every admissible V-module is a direct sum of simple admissible V-module, i.e. every admissible V-module is completely reducible.

Lemma 2.6. *([16, 17]) A vertex operator algebra V is rational, then there are the following results:*

1) each simple admissible V-module is an ordinary V-module;

2) there are only finitely many inequivalent simple modules.

Definition 2.7. ([16]) A vertex operator algebra V is said to be regular if any weak V-module M is a direct sum of simple ordinary V-modules.

By above definitions, we know that a regular vertex operator algebra V is necessarily rational.

2.2 Modules for affine Lie algebras

Let $\mathbf{g}$ be a simple Lie algebra over $\mathbb{C}$ with a triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \eta \oplus \mathbf{n}_+$, where η is the Cartan subalgebra of $\mathbf{g}$. Let Δ be the root system of $(\mathbf{g}, \eta), \Delta_+ \subset \Delta$ the set of positive roots, θ the highest root and $(\cdot, \cdot) : \mathbf{g} \times \mathbf{g} \to \mathbb{C}$ the killing form normalized by the condition $(\theta, \theta) = 2$.

The affine Lie algebra $\widehat{\mathbf{g}}$ associated to $\mathbf{g}$ is the vector space $\mathbf{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ equipped with the usual bracket operation, and the canonical central element c ([10]). Let $\widehat{\mathbf{g}} = \widehat{\mathbf{n}}_- \oplus \widehat{\eta} \oplus \widehat{\mathbf{n}}_+$ be the corresponding triangular decomposition of $\widehat{\mathbf{g}}$, where

$$\widehat{\mathbf{n}}_- = \mathbf{n}_- \otimes 1 \oplus \mathbf{g} \otimes t^{-1}\mathbb{C}[t^{-1}]; \widehat{\mathbf{n}}_+ = \mathbf{n}_+ \otimes 1 \oplus \mathbf{g} \otimes t\mathbb{C}[t]; \widehat{\eta} = \eta \oplus \mathbb{C}c.$$

Definition 2.8. ([10]) A module V is called a highest weight $\mathbf{g}$-module if it contains a weight vector v, such that

$$\mathbf{n}_+ \cdot v = 0, V = U(\mathbf{n}_-)v.$$

Let $\mathcal{B} = \mathbf{n}_+ \oplus \eta$ be the Borel subalgebra of $\mathbf{g}$. For $\lambda \in \eta^*$, we define a $\mathcal{B}$-module $\mathbb{C}v_\lambda$ by $\mathbf{n}_+ \cdot v_\lambda = 0$, and $h \cdot v_\lambda = \lambda(h)v_\lambda$, where $h \in \eta$. Then the Verma module

$$M(\lambda) = U(\mathbf{g}) \otimes_{U(\mathcal{B})} \mathbb{C}v_\lambda \cong U(\mathbf{n}_-) \otimes_{\mathbb{C}} \mathbb{C}v_\lambda = U(\mathbf{n}_-)v_\lambda.$$

Lemma 2.9. *([10]) $M(\lambda)$ has a unique maximal proper submodule $M(\lambda)^{'}$, and $L(\lambda) = M(\lambda)/M(\lambda)^{'}$ is an irreducible highest weight module with the highest weight λ.*

Definition 2.10. ([10]) A weight vector $v \in M(\lambda)$ is called a singular vector if $\mathbf{n}_+ \cdot v = 0$.

Definition 2.11. ([10]) Let V be a $\mathbf{g}$-module, if η acts diagonally on V and Chevalley basis $\{e_i, f_i\}(i = 1, 2, \cdots, l)$ of $\mathbf{g}$ are locally nilpotent on V, then we call V an integrable $\mathbf{g}$-module.

For affine Lie algebra $\widehat{\mathbf{g}}$, let $\widehat{\mathbf{g}}_+ = \mathbf{g} \otimes t\mathbb{C}[t], \widehat{\mathbf{g}}_- = \mathbf{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$, then there is

$$\widehat{\mathbf{g}} = \widehat{\mathbf{g}}_+ \oplus \widehat{\mathbf{g}}_- \oplus \mathbf{g} \oplus \mathbb{C}c.$$

Assume that V is a $\mathbf{g}$-module, we can regard V as a $\widehat{\mathbf{g}}_+ \oplus \mathbf{g} \oplus \mathbb{C}c$-module by $\widehat{\mathbf{g}}_+ \cdot v = 0$, for $v \in V$; c acts as a scalar k on $\mathbb{C}$. Then the induced $\widehat{\mathbf{g}}$-module (generalized Verma module)

$$N(k, V) = Ind^{\widehat{\mathbf{g}}}_{\widehat{\mathbf{g}}_+ \oplus \mathbf{g} \oplus \mathbb{C}c} V = U(\widehat{\mathbf{g}}) \otimes_{U(\widehat{\mathbf{g}}_+ \oplus \mathbf{g} \oplus \mathbb{C}c)} V.$$

For Lie algebra $(\mathbf{g}, \eta)$, we choose a weight $\lambda \in \eta^*$, denote by $V(\lambda)$ the irreducible highest weight $\mathbf{g}$-module with the highest weight λ, hence we have an induced $\widehat{\mathbf{g}}$-module $N(k, \lambda) := N(k, V(\lambda))$.

Let $J(k, \lambda)$ be the maximal proper submodule of $N(k, \lambda)$. Set $L(k, \lambda) = N(k, \lambda)/J(k, \lambda)$. There is

Lemma 2.12. *([15]) $L(k, \lambda)$ is the unique $\widehat{\mathbf{g}}$-module satisfying the following properties:*

1) $L(k, \lambda)$ is irreducible as a $\widehat{\mathbf{g}}$-module;

2) the central element c acts on $L(k, \lambda)$ as $k\mathrm{Id}$;

3) $V_\lambda = \{a \in L(k, \lambda) \mid \widehat{\mathbf{g}}_+ \cdot a = 0\}$ is the irreducible $\mathbf{g}$-module with the highest weight λ.

2.3 Roots and Weights of Affine Lie algebras

Let Δ be the root system of $(\mathbf{g}, \eta)$, and Δ_+ its positive root set. Set $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\} \subset \Delta$ be the simple root set of $(\mathbf{g}, \eta)$ and θ be the highest root. For affine Lie algebra $\widehat{\mathbf{g}}$, there are

$$\widehat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_l\};$$
$$\widehat{\Delta} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}, n \neq 0\};$$
$$\widehat{\Delta}_+ = \{\alpha + n\delta \mid \alpha \in \Delta, n > 0\} \cup \{\alpha \mid \alpha \in \Delta_+\},$$

where $\delta = \sum\limits_{i=0}^{l} a_i\alpha_i$(cf. [10]).

For Lie algebra $\mathbf{g}$, the simple coroot set is $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_l^\vee\} \subset \eta$. If $\{\omega_1, \omega_2, \cdots, \omega_l\}$ are the fundamental weight of $\mathbf{g}$, we know that there are $\omega_i(\alpha_j^\vee) = \delta_{ij}$, for $i, j = 1, 2, \cdots, l$. And $\widehat{\mathbf{g}}$ has the fundamental weights $\{\widehat{\omega}_0, \widehat{\omega}_1, \cdots, \widehat{\omega}_l\}$, where $\widehat{\omega}_0 \in \widehat{\eta}^*$ defined by $\widehat{\omega}_0(\eta) = 0, \widehat{\omega}_0(c) = 1$, and $\widehat{\omega}_i = a_i^\vee \widehat{\omega}_0 + \omega_i (i = 1, 2, \cdots, l)$, so the sum of fundamental weight of $\widehat{\mathbf{g}}$ is

$$\widehat{\rho} = \sum_{i=0}^{l} \widehat{\omega}_i = h^\vee \widehat{\omega}_0 + \sum_{i=1}^{l} \omega_i.$$

Set $\overline{\rho} = \sum\limits_{i=1}^{l} \omega_i$ be sum of fundamental weight of $\mathbf{g}$, then there is $\widehat{\rho} = h^\vee \widehat{\omega}_0 + \overline{\rho}$.

Let $P^{\mathbf{g}}$ be the set of dominant weights for $\mathbf{g}$, then $P^{\mathbf{g}} = \{\lambda \in \eta^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, i = 1, 2, \cdots, l\}$, and the set of dominant integral weights for $\mathbf{g}$ is $P_+^{\mathbf{g}} = \{\lambda \in \eta^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}, i = 1, 2, \cdots, l\} \subset P^{\mathbf{g}}$.

Lemma 2.13. *([10]) An irreducible highest weight $\mathbf{g}$-module $V(\mu)$ is integrable if and only if the highest weight $\mu \in P_+^{\mathbf{g}}$.*

2.4 Vertex operator algebras $N(k, 0)$ and $L(k, 0)$ for $k \neq -h^\vee$

Since $V(0)$ is the 1-dimensional trivial $\mathbf{g}$-module, it can be identified with $\mathbb{C}$. Denote by $\mathbf{1} = 1 \otimes 1 \in N(k, 0)$, then $N(k, 0) = Span_{\mathbb{C}}\{g_1(-n_1 - 1)g_2(-n_2 - 1) \cdots g_m(-n_m - 1) \mid g_1, \cdots, g_m \in \mathbf{g}, n_1, n_2, \cdots, n_m \in \mathbb{N}\}$, where $g(n)$ is denoted by the representation image of $g \otimes t^n$ for $g \in \mathbf{g}, n \in \mathbb{Z}$. By Dong lemma ([9]), the map $Y(\cdot, t) : N(k, 0) \longrightarrow (\mathrm{End} N(k, 0))[[t, t^{-1}]]$ is uniquely determined by

$$Y(\mathbf{1}, t) = \mathrm{Id}_{N(k,0)}; Y(g(-1)\mathbf{1}, t) = \sum_{n \in \mathbb{Z}} g(n) t^{-n-1}, \quad \text{for} \quad g \in \mathbf{g}.$$

In the case that $k \neq -h^\vee$, $N(k, 0)$ has a conformal vector

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathbf{g}} A^i(-1) B^i(-1) \mathbf{1}, \tag{2.7}$$

where $\{A^i \mid i = 1, \cdots, \dim \mathbf{g}\}$ is an arbitrary basis of $\mathbf{g}$, and $\{B^i \mid i = 1, \cdots, \dim \mathbf{g}\}$ the corresponding dual basis of $\mathbf{g}$ with respect to the Killing form $(\cdot, \cdot)$. From [15], we have the following result

Proposition 2.14. *If $k \neq -h^\vee$, $(N(k, 0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra.*

For any $\mu \in \eta, N(k, \mu)$ is an admissible $N(k, 0)$-module. Denote by $v_{k,\mu}$ the highest weight vector of $L(k, \mu)$, then the lowest conformal weight of $L(k, \mu)$ is given by the relation

$$L(0) \cdot v_{k,\mu} = \frac{(\mu, \mu + 2\overline{\rho})}{2(k + h^\vee)} v_{k,\mu}, \tag{2.8}$$

where $\overline{\rho}$ is the sum of fundamental weights of $\mathbf{g}$.

Since every $\widehat{\mathbf{g}}$-submodule of $N(k,0)$ is also an ideal in the vertex operator algebra $N(k,0)$, it follows that $L(k,0)$ is a simple vertex operator algebra, for any $k \neq -h^\vee$.

2.5 Zhu's $A(V)$ theory

Let V be a vertex operator algebra. Following [13], we define bilinear maps $* : V \times V \to V$ and $\circ : V \times V \to V$ as follows. For any homogeneous element $a \in V$ and for any $b \in V$

$$a \circ b = \mathrm{Res}_t \frac{(1+t)^{wta}}{t^2} Y(a,t)b \tag{2.9}$$

$$a * b = \mathrm{Res}_t \frac{(1+t)^{wta}}{t} Y(a,t)b \tag{2.10}$$

and extend to $V \times V \to V$ by linearity, where wta is the weight of a. Denote by $O(V) = \mathrm{Span}_{\mathbb{C}}\{a \circ b \mid a, b \in V\}$, and by $A(V) = V/O(V)$. For $a \in V$, denote by $[a]$ the image of a under the projection of V onto $A(V)$. The multiplication $*$ induces the multiplication on $A(V)$ and such that $A(V)$ has a structure of associative algebra.

Proposition 2.15. *([15]) Let I be an ideal of V. Assume $\mathbf{1} \notin I, \omega \notin I$, then $A(V/I)$ is isomorphic to $A(V)/A(I)$, where $A(I)$ is the image of I in $A(V)$.*

Proposition 2.16. *([15]) The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathbf{g})$. The isomorphism $F : A(N(k,0)) \longrightarrow U(\mathbf{g})$ is given by*

$$F([g_1(-n_1-1)g_2(-n_2-1)\cdots g_m(-n_m-1)\mathbf{1}]) = (-1)^{\sum\limits_{i=1}^{m} n_i} g_1 g_2 \cdots g_m, \tag{2.11}$$

for any $g_1, g_2, \cdots, g_m \in \mathbf{g}$, and any $n_1, n_2, \cdots, n_m \in \mathbb{N}$.

Proposition 2.17. *([15]) Assume that the maximal $\widehat{\mathbf{g}}$-submodule of $N(k,0)$ is generated by a singular vector v, i.e. $J(k,0) = U(\widehat{\mathbf{g}})v$, then*

$$A(L(k,0)) \cong U(\mathbf{g})/I, \tag{2.12}$$

where I is the two-side ideal of $U(\mathbf{g})$ generated by $u = F([v])$).

Theorem 2.18. *([15]) Let k be a positive integer, then*

$$U(\mathbf{g})/\langle e_\theta^{k+1} \rangle \cong A(L(k,0)), \tag{2.13}$$

where e_θ is an element in the root space $\mathbf{g}_\theta$ of the highest root θ, and $\langle e_\theta^{k+1}\rangle$ is the two sided ideal generated by e_θ^{k+1}.

Theorem 2.19. *([15]) If $k \in \mathbb{Z}_+$, then the vertex operator algebra $L(k,0)$ is rational. The set*

$$\{L(k,\mu) \mid k \in \mathbb{Z}_+, \mu \in \eta^* \textit{ is an integrable weight satisfying } \langle \mu,\theta\rangle \leq k\} \tag{2.14}$$

provides a complete list of simple admissible $L(k,0)$-module.

Proposition 2.20. *([15]) Let $k \in \mathbb{Z}_+$, the maximal proper submodule $J(k,0)$ of $N(k,0)$ is generated by $e_\theta(-1)^{k+1}\mathbf{1}$, and $e_\theta(-1)^{k+1}\mathbf{1}$ is a singular vector for $\widehat{\mathbf{g}}$ in $N(k,0)$.*

3 Lie algebra $\mathbf{g}_{E_8}$ and $\mathbf{g}_{D_8}$

Let $\mathbb{R}^8$ be the 8-dimensional Euclid space, and $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_8\}$ is the orthonormal basis with form as $(0,0,\cdots,i,0,\cdots,0)$, there is root system of Lie algebra $\mathbf{g}_{E_8}$

$$\Delta_{E_8} = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq 8\} \cup \left\{\pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_8)\right\}_{\text{number of minus signs is even}}.$$

The positive root set is

$$\Delta_{E_8}^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 8\} \cup \left\{\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_8)\right\}_{\text{number of minus signs is even}},$$

and we take the simple roots

$$\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \cdots, \alpha_6 = \epsilon_7 - \epsilon_8,$$
$$\alpha_7 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - \epsilon_2 - \cdots - \epsilon_7), \alpha_8 = \epsilon_7 + \epsilon_8.$$

$\theta = \epsilon_1 + \epsilon_2 = \sum_{i=1}^{5}(i+1)\alpha_i + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ is the highest root, and number of positive roots $|\Delta_{E_8}^+| = 120$, the dual Coxter number $h_{E_8}^\vee = 30$. As a vector space the dimension of $\mathbf{g}_{E_8}$ is 248. The corresponding fundamental weights are

$$\omega_1 = \epsilon_1 + \epsilon_2, \omega_2 = 2\epsilon_1 + \epsilon_2 + \epsilon_3, \cdots, \omega_5 = 5\epsilon_1 + \epsilon_2 + \cdots + \epsilon_6,$$
$$\omega_6 = \frac{1}{2}(7\epsilon_1 + \epsilon_2 + \cdots + \epsilon_7 - \epsilon_8), \omega_7 = 2\epsilon_1, \omega_8 = \frac{1}{2}(5\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8).$$

We assume that $\{h_i, e_i, f_i \mid i = 1, 2, \cdots, 8\}$ are the Chevalley generators of $\mathbf{g}_{E_8}$. Then all the other root vectors can be fixed by the following relations(cf. [10])

$$[e_\alpha, e_\beta] = e_{\alpha+\beta}; [f_\alpha, f_\beta] = -f_{\alpha+\beta},$$

where $\alpha, \beta, \alpha+\beta \in \Delta^+_{E_8}$. Moreover, for $\alpha, \beta, \beta-\alpha \in \Delta^+_{E_8}$, they can be chosen to satisfy the following

$$[f_\alpha, e_\beta] = e_{\beta-\alpha}; [e_\alpha, f_\beta] = -f_{\beta-\alpha}.$$

Denote by $h_\alpha = \alpha^\vee = [e_\alpha, f_\alpha]$, for any $\alpha \in \Delta^+_{E_8}$.

For Lie algebra $\mathbf{g}_{D_8} = \mathbf{so}(16, \mathbb{C})$, we take the root system and positive root set, respectively.

$$\Delta_{D_8} = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq 8\};$$
$$\Delta^+_{D_8} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq 8\},$$

also we can take the simple roots

$$\beta_1 = \epsilon_1 - \epsilon_2, \beta_2 = \epsilon_2 - \epsilon_3, \cdots, \beta_7 = \epsilon_7 - \epsilon_8, \beta_8 = \epsilon_7 + \epsilon_8.$$

The highest root $\theta = \epsilon_1 + \epsilon_2 = \beta_1 + 2\sum\limits_{i=2}^{6} \beta_i + \beta_7 + \beta_8$, and number of positive root set is $|\Delta^+_{D_8}| = 56$, the dual Coxter number $h^\vee_{D_8} = 14$. As a vector space the dimension of $\mathbf{g}_{D_8}$ is 120. The corresponding fundamental weights are

$$\overline{\omega}_1 = \epsilon_1, \overline{\omega}_2 = \epsilon_1 + \epsilon_2, \cdots, \overline{\omega}_6 = \sum_{i=1}^{6} \epsilon_i, \overline{\omega}_7 = \frac{1}{2}\left(\sum_{i=1}^{7} \epsilon_i - \epsilon_8\right), \overline{\omega}_8 = \frac{1}{2}\left(\sum_{i=1}^{8} \epsilon_i\right).$$

4 Vertex operator algebra $L_{D_8}(k, 0)$ and $L_{E_8}(k, 0)$ for $k \in \mathbb{Z}_+$

For Lie algebra $\mathbf{g}_{D_8}$, if $\mu = \sum_{i=1}^{8} c_i \overline{\omega}_i \in \mathcal{P}^{D_8}_+$, then it requires that $\langle \mu, \beta_j^\vee \rangle = \langle \sum_{i=1}^{8} c_i \overline{\omega}_i, \beta_j^\vee \rangle = c_i \delta_{ij} \in \mathbb{N}$, i.e. $c_i \in \mathbb{N}, \quad i = 1, 2, \cdots, 8$. By Theorem 2.19, $L(k, \mu)$ is a simple admissible $L(k, 0)$-module, if and only if μ satisfies the condition

$$\begin{cases} \langle \mu, \theta \rangle \leq k; \\ c_i \in \mathbb{N}, \quad i = 1, 2, \cdots, 8. \end{cases} \tag{4.1}$$

Since the highest root $\theta = \epsilon_1 + \epsilon_2$, the condition (4.1) is equivalent to the condition

$$\begin{cases} c_1 + \sum\limits_{i=2}^{6} 2c_i + c_7 + c_8 \leq k; \\ c_i \in \mathbb{N}, i = 1, 2, \cdots, 8. \end{cases} \tag{4.2}$$

Hence we have the following result

Corollary 4.1. *If $k = 1$, there is a complete list of simple admissible $L_{D_8}(1,0)$-module*

$$\{L_{D_8}(1,0), L_{D_8}(1,\overline{\omega}_1), L_{D_8}(1,\overline{\omega}_7), L_{D_8}(1,\overline{\omega}_8)\}. \tag{4.3}$$

For Lie algebra $\mathbf{g}_{E_8}$, let $\lambda = \sum_{i=1}^{8} b_i\omega_i$, if $L_{E_8}(k,\lambda)$ is a simple admissible $L_{E_8}(1,0)$-module, it must satisfy the condition from Theorem 2.19

$$\begin{cases} 2b_1 + 3b_2 + 4b_3 + 5b_4 + 6b_5 + 4b_6 + 2b_7 + 3b_8 \le k; \\ b_i \in \mathbb{N}, i = 1, 2, \cdots, 8. \end{cases} \tag{4.4}$$

So we have

Corollary 4.2. *If $k = 1$, the simple admissible $L_{E_8}(1,0)$-module is only $L_{E_8}(1,0)$ itself.*

5 Vertex operator algebra $L_{D_8}(1,0)$ and $L_{E_8}(1,0)$

In this section we shall give our constructions of the embedding vertex operator algebras $L_{D_8}(1,0)$ into $L_{E_8}(1,0)$.

For vertex operator algebra $L_{D_8}(1,0)$, we choose $\alpha \in \Delta^+_{D_8}$ so that $\left\{\frac{1}{\sqrt{2}}h_\alpha\right\}$ is the orthonormal basis of η_{D_8} with respect to the killing form $(\cdot,\cdot)$. Denote such root set by $\underline{\Delta}^+_{D_8}$. It's known that $|\underline{\Delta}^+_{D_8}| = 8$. By Segal-Sugawara construction,

$$\omega_{D_8} = \frac{1}{30}(\frac{1}{2}\sum_{\alpha\in\underline{\Delta}^+_{D_8}} h^2_\alpha(-1) + \sum_{\alpha\in\Delta^+_{D_8}} (e_\alpha(-1)f_\alpha(-1) + f_\alpha(-1)e_\alpha(-1))) \tag{5.1}$$

is one of conformal vectors of vertex operator algebra $L_{D_8}(1,0)$.

Since $\Delta_{D_8} \subset \Delta_{E_8}$, and $\dim\eta_{E_8} = 8$, so $\left\{\frac{1}{\sqrt{2}}h_\alpha \mid \alpha \in \underline{\Delta}^+_{D_8}\right\}$ can be chosen as an orthonormal basis of η_{E_8} with respect to the killing form $(\cdot,\cdot)$. By Segal-Suganara construction, we know

$$\omega_{E_8} = \frac{1}{62}(\frac{1}{2}\sum_{\alpha\in\underline{\Delta}^+_{D_8}} h^2_\alpha(-1) + \sum_{\alpha\in\Delta^+_{E_8}} (e_\alpha(-1)f_\alpha(-1) + f_\alpha(-1)e_\alpha(-1))) \tag{5.2}$$

is a conformal vector for vertex operator algebra $L_{E_8}(1,0)$. Here, we fix a choice $\underline{\Delta}^+_{D_8} = \{\epsilon_1 \pm \epsilon_2, \epsilon_3 \pm \epsilon_4, \epsilon_5 \pm \epsilon_6, \epsilon_7 \pm \epsilon_8\}$ for conveniences.

By Proposition 2.20, there is the following results

Proposition 5.1. $L_{E_8}(1,0) = N_{E_8}(1,0)/J_{E_8}(1,0)$ *is a simple vertex operator algebra, where* $J_{E_8}(1,0)$ *is generated by the singular vector* $v_{E_8} = e^2_\theta(-1)\mathbf{1}$, *i.e.* $J_{E_8}(1,0) = U(\widehat{\mathbf{g}}_{E_8})v_{E_8}$; $L_{D_8}(1,0) = N_{D_8}(1,0)/J_{D_8}(1,0)$ *is a simple vertex operator algebra, where* $J_{D_8}(1,0)$ *is generated by the singular vector* $v_{D_8} = e^2_\theta(-1)\mathbf{1}$, *i.e.* $J_{D_8}(1,0) = U(\widehat{\mathbf{g}}_{D_8})v_{D_8}$.

Since $v_{D_8} = v_{E_8}$ and $\mathbf{g}_{D_8}$ is a Lie subalgebra of $\mathbf{g}_{E_8}$, so we know that

Proposition 5.2. *$L_{D_8}(1,0)$ is a vertex subalgebra of $L_{E_8}(1,0)$.*

Since $\mathbf{g}_{D_8}$ can embed into $\mathbf{g}_{E_8}$ as a Lie subalgebra, in the following we show $L_{D_8}(1,0)$ can embed into $L_{E_8}(1,0)$ as a vertex operator subalgebra.

Lemma 5.3. *Let $(V, Y, \mathbf{1}, \omega_V)$ be a vertex operator algebra, $U \subset V$ is a vertex subalgebra of V, and $(U, Y, \mathbf{1}, \omega_U)$ itself is a vertex operator algebra, then $(V, Y, \mathbf{1}, \omega_V)$ is a weak U-module.*

Proof. Since V is a vertex operator algebra and $U \subset V$ as a vertex subalgebra, we define

$$
\begin{aligned}
&Y_U : U \longrightarrow (\mathrm{End} V)[[t, t^{-1}]] \\
&u \longmapsto Y_U(u,t) = Y(u,t) = \sum_{n\in\mathbb{Z}} u_n t^{-n-1},
\end{aligned}
$$

and for any $u, v \in U, w \in V$, the map Y_U satisfies that

$$u_n w = 0, \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large.} \tag{5.3}$$

$$Y_U(\mathbf{1}, t) = Id_V; \tag{5.4}$$

$$
\begin{aligned}
t_0^{-1}\delta\left(\frac{t_1 - t_2}{t_0}\right) Y_U(u,t_1)Y_U(v,t_2) - t_0^{-1}\delta\left(\frac{t_2 - t_1}{-t_0}\right) Y_U(v,t_2)Y_U(u,t_1) \\
= t_2^{-1}\delta\left(\frac{t_1 - t_0}{t_2}\right) Y_U(Y(u,t_0)v, t_2).
\end{aligned} \tag{5.5}
$$

So (V, Y) is a weak U-module for vertex operator algebra U.

By lemma 2.1, there is

Lemma 5.4. *Assume that U, V are the same as above proposition. If vertex operator algebra U has central charge c_U, the following relations hold on V*

$$[L_U(m), L_U(n)] = (m-n)L_U(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_U, \tag{5.6}$$

for $m, n \in \mathbb{Z}$, where

$$L_U(n) = \omega_{U(n+1)}, \quad i.e.\ Y_U(\omega, t) = \sum_{n\in\mathbb{Z}} L_U(n) t^{-n-2}.$$

$$\frac{d}{dz} Y_U(v,t) = Y_U(L_U(-1)v, t). \tag{5.7}$$

If U is a regular vertex operator algebra, we know it has finitely many simple U-modules. Since V is a weak U-module, then it can be written as direct sum of these simple U-modules. So we know that $L_U(0)$ acts semisimply on V.

Lemma 5.5. *Assume that $(V, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra, then the vertex operator map $Y : V \longrightarrow (EndV)[[t, t^{-1}]]$ is injective.*

Proof. Denote the kernel of the vertex operator map Y by $KerY$. It is easy to check that $KerY$ is an ideal of V. Since V is simple, then V has only ideal 0 and V itself. and because $\mathbf{1} \notin KerY$, we know $KerY = 0$, so Y is injective.

By Proposition 5.2, we know $L_{D_8}(1,0)$ is a vertex subalgebra of $L_{E_8}(1,0)$, and $L_{D_8}(1,0)$ is a vertex operator algebra with the conformal vector ω_{D_8}. According to Proposition 5.3, $L_{E_8}(1,0)$ is a weak $L_{D_8}(1,0)$-module, and $\omega_{D_8} \in L_{E_8}(1,0)$ satisfies the relations (5.6) and (5.7). We also know $L_{D_8}(0)$ acts semisimply on $L_{E_8}(1,0)$, where

$$Y(\omega_{D_8}, t) = \sum_{n\in\mathbb{Z}} \omega_{D_8(n)} t^{-n-1} = \sum_{n\in\mathbb{Z}} L_{D_8}(n) t^{-n-2}. \tag{5.8}$$

For the conformal vector ω_{E_8} of $L_{E_8}(1,0)$, denote by

$$Y(\omega_{E_8}, t) = \sum_{n\in\mathbb{Z}} L_{E_8}(n) t^{-n-2}. \tag{5.9}$$

The constructions of $L_{D_8}(1,0), L_{E_8}(1,0)$ imply that the action of $L_{E_8}(0)$ on subalgebra $L_{D_8}(1,0)$ is the same as that of $L_{D_8}(0)$. To prove $L_{E_8}(0) = L_{D_8}(0)$ on $L_{E_8}(1,0)$, we need to the following several lemmas.

Here, we have

$$\Delta^+_{E_8} \backslash \Delta^+_{D_8} = \left\{ \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_8) \right\}_{\text{sum of minus sign is even}}$$
$$\underline{\Delta}^+_{D_8} = \{\epsilon_1 \pm \epsilon_2, \epsilon_3 \pm \epsilon_4, \epsilon_5 \pm \epsilon_6, \epsilon_7 \pm \epsilon_8\}.$$

Lemma 5.6. *For any $\alpha' \in \Delta^+_{E_8} \backslash \Delta^+_{D_8}$, there is*

$$\frac{1}{60} \sum_{\alpha\in\underline{\Delta}^+_{D_8}} \langle \alpha', \alpha \rangle^2 e_{\alpha'}(-1)\mathbf{1} = \frac{1}{15} e_{\alpha'}(-1)\mathbf{1}. \tag{5.10}$$

Proof. In $\mathbb{R}^8$, there is inner product $(\cdot,\cdot)$, we know

$$\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, \quad \forall \alpha, \beta \in \mathbb{R}^8.$$

So for any $\alpha' \in \Delta^+_{E_8} \backslash \Delta^+_{D_8}$, $(\alpha', \alpha') = 2$, hence $\langle \alpha', \alpha \rangle = (\alpha', \alpha)$ for $\alpha \in \underline{\Delta}^+_{D_8}$. By the orthonormality of the basis $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_8\}$, there is

$$\sum_{\alpha\in\underline{\Delta}^+_{D_8}} \langle \alpha', \alpha \rangle^2 = 4.$$

Lemma 5.7. *For any $\alpha' \in \Delta_{E_8}^+ \backslash \Delta_{D_8}^+$, there is*

$$\frac{1}{30}\sum_{\alpha\in\Delta_{D_8}^+}(\langle\alpha',\alpha\rangle e_{\alpha'}(-1)\mathbf{1}+2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1})=\frac{14}{15}e_{\alpha'}(-1)\mathbf{1}. \tag{5.11}$$

Proof. According to the relations between $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ in $\mathbb{R}^8$, if $\alpha' \in \Delta_{E_8}^+ \backslash \Delta_{D_8}^+$, and the sum of minus sign is 0, then there are

$$\sum_{\alpha\in\Delta_{D_8}^+}\langle\alpha',\alpha\rangle=\sum_{i=1}^{7}i=28. \tag{5.12}$$

Since $\alpha'+\alpha\notin\Delta_{E_8}^+$, then $[f_\alpha,[e_\alpha,e_{\alpha'}]]=0$, hence (5.11) holds.

If $\alpha' \in \Delta_{E_8}^+ \backslash \Delta_{D_8}^+$, and the sum of minus sign is 2. Let

$$\alpha'=\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots-\epsilon_i+\cdots-\epsilon_j+\cdots+\epsilon_8),\quad 2\le i<j\le 8,$$

then there are

$$\sum_{\alpha\in\Delta_{D_8}^+}\langle\alpha',\alpha\rangle=7+\cdots-(8-i)+\cdots-(8-j)+\cdots+1=2(i+j)-4.$$

And $|\{\alpha \mid \alpha\in\Delta_{D_8}^+,\alpha'+\alpha\in\Delta_{E_8}^+\}|=16-(i+j)$, then there is

$$\begin{aligned}&\sum_{\alpha\in\Delta_{D_8}^+}\langle\alpha',\alpha\rangle e_{\alpha'}(-1)\mathbf{1}+2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1}\\&\qquad=(2(i+j)-4+32-2(i+j))e_{\alpha'}(-1)\mathbf{1}=28e_{\alpha'}(-1)\mathbf{1}.\end{aligned}$$

If $\alpha' \in \Delta_{E_8}^+ \backslash \Delta_{D_8}^+$, and the sum of minus sign is 4. Let

$$\alpha'=\frac{1}{2}(\epsilon_1-\cdots+\epsilon_i-\cdots+\epsilon_j-\cdots+\epsilon_s-\cdots-\epsilon_8),\quad 2\le i<j<s\le 8,$$

then we have

$$\begin{aligned}\sum_{\alpha\in\Delta_{D_8}^+}\langle\alpha',\alpha\rangle&=7-\cdots-(8-i+1)+(8-i)-\cdots-(8-j+1)+(8-j)\\&\quad-\cdots-(8-s+1)+(8-s)-\cdots-1\\&=34-2(i+j+s),\end{aligned}$$

$$\begin{aligned}\sum_{\alpha\in\Delta_{D_8}^+}2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1}&=2(28-7-(8-i)-(8-j)-(8-s))e_{\alpha'}(-1)\mathbf{1}\\&=(2(i+j+s)-6)e_{\alpha'}(-1)\mathbf{1},\end{aligned}$$

then there is

$$\sum_{\alpha\in\Delta^+_{D_8}}(\langle\alpha',\alpha\rangle e_{\alpha'}(-1)\mathbf{1}+2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1})=28e_{\alpha'}(-1)\mathbf{1},$$

hence (5.11) holds.

If $\alpha'\in\Delta^+_{E_8}\backslash\Delta^+_{D_8}$, and the sum of minus sign is 6. Let

$$\alpha'=\frac{1}{2}(\epsilon_1-\epsilon_2-\cdots+\epsilon_i-\cdots-\epsilon_8),\quad 2\le i\le 8,$$

then there are

$$\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha',\alpha\rangle=7-\cdots-(8-i+)+(8-i)-\cdots-1=2-2i;$$

and

$$\sum_{\alpha\in\Delta^+_{D_8}}2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1}=2(28-7-(8-i))=26+2i.$$

Hence we get

$$\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha',\alpha\rangle e_{\alpha'}(-1)\mathbf{1}+2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1}=28e_{\alpha'}(-1)\mathbf{1}.$$

Lemma 5.8. $\alpha'\in\Delta^+_{E_8}\backslash\Delta^+_{D_8}$, *there is*

$$\frac{1}{30}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}+\frac{1}{15}\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}=h_{\alpha'}(-1)\mathbf{1}\qquad(5.13)$$

Proof. Note $h_\alpha=\alpha^\vee=\frac{2\alpha}{(\alpha,\alpha)}$. Since $(\alpha,\alpha)=2$ for $\alpha\in\Delta_{E_8}$, then $\alpha=\alpha^\vee,\forall\alpha\in\Delta_{E_8}$.

If $\alpha'\in\Delta^+_{E_8}\backslash\Delta^+_{D_8}$, and the sum of minus sign is 0. Let $\alpha'=\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)$, there are

$$\frac{1}{30}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}+\frac{1}{15}\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}$$

$$= \frac{1}{30}(h_{\epsilon_1+\epsilon_2} + h_{\epsilon_3+\epsilon_4} + h_{\epsilon_5+\epsilon_6} + h_{\epsilon_7+\epsilon_8})$$
$$+ \frac{1}{15}(\sum_{i=2}^{8} h_{\epsilon_1+\epsilon_i} + \sum_{i=3}^{8} h_{\epsilon_2+\epsilon_i} + \cdots + h_{\epsilon_7+\epsilon_8})$$
$$= (\frac{1}{30}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8)$$
$$+ \frac{1}{15}(\sum_{i=2}^{8}(\epsilon_1 + \epsilon_i) + \sum_{i=3}^{8}(\epsilon_2 + \epsilon_i) + \cdots + \epsilon_7 + \epsilon_8))(-1)\mathbf{1}$$
$$= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)(-1)\mathbf{1}$$
$$= h_{\alpha'}(-1)\mathbf{1}.$$

If $\alpha' \in \Delta_{E_8}^{+} \backslash \Delta_{D_8}^{+}$, and the sum of minus sign is 2. Let

$$\alpha' = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots - \epsilon_i + \cdots - \epsilon_j + \cdots + \epsilon_8), \quad 2 \leq i < j \leq 8,$$

then there are

$$\frac{1}{30} \sum_{\alpha \in \underline{\Delta}_{D_8}^{+}} \langle \alpha, \alpha' \rangle h_\alpha(-1)\mathbf{1} + \frac{1}{15} \sum_{\alpha \in \Delta_{D_8}^{+}} \langle \alpha, \alpha' \rangle h_\alpha(-1)\mathbf{1}$$
$$= \frac{1}{30}(\epsilon_1 + \epsilon_2 + \cdots - \epsilon_i + \cdots - \epsilon_j + \cdots + \epsilon_8) + \frac{1}{15}(\sum_{l=2, l\neq i,j}^{8} h_{\epsilon_1+\epsilon_l} + h_{\epsilon_1-\epsilon_i}$$
$$+ h_{\epsilon_1-\epsilon_j} + \cdots + \sum_{l=i-1, l\neq j}^{8} h_{\epsilon_{i-1}+\epsilon_l} + h_{\epsilon_{i-1}+\epsilon_i} - \sum_{l=i+1, l\neq j}^{8} h_{\epsilon_i-\epsilon_l} - h_{\epsilon_i+\epsilon_j} + \cdots$$
$$+ \sum_{l=j+1}^{8} h_{\epsilon_{j-1}+\epsilon_l} + h_{\epsilon_{j-1}+\epsilon_j} - \sum_{l=j+1}^{8} h_{\epsilon_j-\epsilon_l} + \cdots + h_{\epsilon_7+\epsilon_8})(-1)\mathbf{1}$$
$$= \frac{1}{30}(\epsilon_1 + \epsilon_2 + \cdots - \epsilon_i + \cdots - \epsilon_j + \cdots + \epsilon_8))(-1)\mathbf{1}$$
$$+ \frac{7}{15}(\epsilon_1 + \epsilon_2 + \cdots - \epsilon_i + \cdots - \epsilon_j + \cdots + \epsilon_8))(-1)\mathbf{1}$$
$$= h_{\alpha'}(-1)\mathbf{1}.$$

In the cases that the sum of minus sign of $\alpha' \in \Delta_{E_8}^{+} \backslash \Delta_{D_8}^{+}$ is 4 and 6, it is easy to check that (5.13) holds as similar way to above two cases. Finally, we have shown the lemma.

As similar to above three lemmas, we also get the following two lemmas

Lemma 5.9. *For any $\alpha' \in \Delta_{E_8}^{+} \backslash \Delta_{D_8}^{+}$, there is*

$$\frac{1}{60} \sum_{\alpha \in \underline{\Delta}_{D_8}^{+}} \langle \alpha', \alpha \rangle^2 f_{\alpha'}(-1)\mathbf{1} = \frac{1}{15} f_{\alpha'}(-1)\mathbf{1}.$$

Lemma 5.10. *For any $\alpha' \in \Delta^+_{E_8}\backslash\Delta^+_{D_8}$, there is*

$$\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\langle\alpha',\alpha\rangle f_{\alpha'}(-1)\mathbf{1}+2[e_\alpha,[f_\alpha,f_{\alpha'}]](-1)\mathbf{1})=\frac{14}{15}f_{\alpha'}(-1)\mathbf{1}.$$

By above some lemmas, we have the following conclusion

Proposition 5.11. *As operators of vertex operator algebra $L_{E_8}(1,0)$, there is $L_{D_8}(0)=L_{E_8}(0)$ on $L_{E_8}(1,0)$.*

Proof. From above statement, we have known $L_{D_8}(0)=L_{E_8}(0)$ on $L_{D_8}(1,0)$, so we only need to show that $L_{D_8}(0)=L_{E_8}(0)$ on $L_{E_8}(1,0)\backslash L_{D_8}(1,0)$.

Since elements $\{e_\alpha(-1)\mathbf{1}, f_\alpha(-1)\mathbf{1}, h_\alpha(-1)\mathbf{1} \mid \alpha\in\Delta^+_{E_8}\}$ generate the vertex operator algebra $L_{E_8}(1,0)$, hence it is sufficient to check that $L_{D_8}(0)=L_{E_8}(0)$ on $\{e_\alpha(-1)\mathbf{1}, f_\alpha(-1)\mathbf{1}, h_\alpha(-1)\mathbf{1} \mid \alpha\in\Delta^+_{E_8}\backslash\Delta^+_{D_8}\}$.

By (5.1), we have

$$\begin{aligned}L_{D_8}(0)=&\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{m\in\mathbb{Z}}:h_\alpha(m)h_\alpha(-m):)+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{m\in\mathbb{Z}}:e_\alpha(m)f_\alpha(-m):\\&+:f_\alpha(m)e_\alpha(-m):),\end{aligned}\tag{5.14}$$

where $:\cdots:$ is normal order product.

According to the definition of normal order product $:\cdots:$, there is

$$\begin{aligned}L_{D_8}(0)=&\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(2\sum_{m>0}h_\alpha(-m)h_\alpha(m)+h^2_\alpha(0))+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(2\sum_{m>0}\\&(f_\alpha(-m)e_\alpha(m)+e_\alpha(-m)f_\alpha(m))+e_\alpha(0)f_\alpha(0)+f_\alpha(0)e_\alpha(0)).\end{aligned}\tag{5.15}$$

For $\alpha'\in\Delta^+_{E_8}\backslash\Delta^+_{D_8}$, there are

$$\begin{aligned}L_{D_8}(0)\cdot e_{\alpha'}(-1)\mathbf{1}=&\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\langle\alpha,\alpha'\rangle^2 e_{\alpha'}(-1)\mathbf{1}\\&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(e_\alpha(0)f_\alpha(0)+f_\alpha(0)e_\alpha(0))e_{\alpha'}(-1)\mathbf{1}\\=&\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\langle\alpha,\alpha'\rangle^2 e_{\alpha'}(-1)\mathbf{1}\\&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\langle\alpha',\alpha\rangle e_{\alpha'}(-1)\mathbf{1}+2[f_\alpha,[e_\alpha,e_{\alpha'}]](-1)\mathbf{1}),\end{aligned}$$

Using Lemma 5.6 and 5.7, we have

$$L_{D_8}(0)\cdot e_{\alpha'}(-1)\mathbf{1}=\frac{1}{15}e_{\alpha'}(-1)\mathbf{1}+\frac{14}{15}e_{\alpha'}(-1)\mathbf{1}=e_{\alpha'}(-1)\mathbf{1},$$

and

$$\begin{aligned}L_{D_8}(0)\cdot h_{\alpha'}(-1)\mathbf{1}=&\frac{1}{30}(\sum_{\alpha\in\Delta^+_{D_8}}h_\alpha(-1)h_\alpha(1)\\&+\sum_{\alpha\in\Delta^+_{D_8}}(e_\alpha(0)f_\alpha(0)+f_\alpha(0)e_\alpha(0)))h_{\alpha'}(-1)\mathbf{1}\\=&\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}+\frac{1}{15}\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha,\alpha'\rangle h_\alpha(-1)\mathbf{1}.\end{aligned}$$

By Lemma 5.8, we have

$$L_{D_8}(0)\cdot h_{\alpha'}(-1)\mathbf{1}=h_{\alpha'}(-1)\mathbf{1}.$$

Similarly, there is

$$\begin{aligned}L_{D_8}(0)\cdot f_{\alpha'}(-1)\mathbf{1}=&\frac{1}{60}\sum_{\alpha\in\Delta^+_{D_8}}\langle\alpha,\alpha'\rangle^2 f_{\alpha'}(-1)\mathbf{1}\\&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\langle\alpha',\alpha\rangle f_{\alpha'}(-1)\mathbf{1}+2[e_\alpha,[f_\alpha,f_{\alpha'}]](-1)\mathbf{1}),\end{aligned}$$

By Lemma 5.9 and 5.10, we obtain

$$L_{D_8}(0)\cdot f_{\alpha'}(-1)\mathbf{1}=f_{\alpha'}(-1)\mathbf{1}.$$

According to above calculus, we know that $L_{D_8}(0)$ is a gradation operator of $L_{E_8}(1,0)$. Since ω_{E_8} is a conformal vector of vertex operator algebra $L_{E_8}(1,0)$, so $L_{E_8}(0)$ is a gradation operator of $L_{E_8}(1,0)$. By the structure of $L_{E_8}(1,0)$, we know that $L_{D_8}(0)$ and $L_{E_8}(0)$ give the same $\mathbb{N}$-graded structure of $L_{E_8}(1,0)$, hence there holds $L_{D_8}(0)=L_{E_8}(0)$ on $L_{E_8}(1,0)$.

From proposition 5.11, we know that $\omega_{E_8},\omega_{D_8}$ are both conformal vectors of vertex operator algebra $L_{E_8}(1,0)$, and the central charges are respectively

$$c_E=\frac{k\mathrm{dim}\mathbf{g}_{E_8}}{h^\vee_{E_8}+k},c_D=\frac{k\mathrm{dim}\mathbf{g}_{D_8}}{h^\vee_{D_8}+k}.$$

It is possible that $\omega_{E_8}=\omega_{D_8}$ if $c_D=c_E$. Solve the condition that $c_D=c_E$, we get $k=1$, and there are $c_D=c_E=8$, which is a main reason we consider the case of $k=1$. As operators of $L_{E_8}(1,0)$, there is also $L_{D_8}(-1)=L_{E_8}(-1)$. Next we show $\omega_{E_8}=\omega_{D_8}$. At the first, we have

Proposition 5.12. *As vertex operators of vertex operator algebra $L_{E_8}(1,0)$, then there is*

$$Y(\omega_{E_8},t)=Y(\omega_{D_8},t). \tag{5.16}$$

Proof. Since ω_{E_8} is the conformal vector of vertex operator algebra $L_{E_8}(1,0)$ by Segal-Sugawara construction, so for any $A(n):=A\otimes t^n\in\widehat{E}_8$, there is the relation(cf. [9])

$$[L_{E_8}(m),A(n)]=-nA(m+n). \tag{5.17}$$

For the conformal vectors ω_{D_8} and ω_{E_8}, there are

$$\begin{aligned}L_{D_8}(n)=&\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{i\in\mathbb{Z}}:h_\alpha(i)h_\alpha(n-i):)+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}:e_\alpha(i)f_\alpha(n-i):\\&+:f_\alpha(i)e_\alpha(n-i):),\end{aligned} \tag{5.18}$$

$$\begin{aligned}L_{E_8}(n)=&\frac{1}{124}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{i\in\mathbb{Z}}:h_\alpha(i)h_\alpha(n-i):)+\frac{1}{62}\sum_{\alpha\in\Delta^+_{E_8}}(\sum_{i\in\mathbb{Z}}:e_\alpha(i)f_\alpha(n-i):\\&+:f_\alpha(i)e_\alpha(n-i):).\end{aligned} \tag{5.19}$$

Next we compute the relation $[L_{E_8}(m),L_{D_8}(n)]$ for $m,n\in\mathbb{Z}$. It has two cases. We only give detail of case $m>0$. By the similar method, one can get the case $m<0$.

We do it for the following steps.

1)

$$\begin{aligned}&[L_{E_8}(m),\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\sum_{i\in\mathbb{Z}}:h_\alpha(i)h_\alpha(n-i):]\\&=\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{i<n}((-i)h_\alpha(i+m)h_\alpha(n-i)+(i-n)h_\alpha(i)h_\alpha(m+n-i))\\&+\sum_{i>n}((i-n)h_\alpha(m+n-i)h_\alpha(i)-ih_\alpha(n-i)h_\alpha(m+i))-nh_\alpha(m+n)h_\alpha(0))\\&=\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{i\in\mathbb{Z}}((-i):h_\alpha(i+m)h_\alpha(n-i):+\sum_{i\in\mathbb{Z}}(i-n):h_\alpha(i)h_\alpha(m+n-i):\\&+\sum_{i=n+1}^{m+n}(i-n)h_\alpha(m+n-i)h_\alpha(i)-\sum_{i=n}^{m+n-1}(i-n)h_\alpha(i)h_\alpha(m+n-i)\\&-mh_\alpha(m+n)h_\alpha(0))\end{aligned}$$

$$= \frac{1}{60}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}((-i):h_\alpha(i+m)h_\alpha(n-i):+(i-n):h_\alpha(i)h_\alpha(m+n-i):)$$

$$+ m(h_\alpha(0)h_\alpha(m+n)-h_\alpha(m+n)h_\alpha(0)) + \sum_{n<i<m+n}(i-n)[h_\alpha(m+n-i),h_\alpha(i)])$$

$$= \frac{1}{60}\sum_{\alpha\in\Delta^+_{D_8}}\sum_{i\in\mathbb{Z}}((-i):h_\alpha(i+m)h_\alpha(n-i):+(i-n):h_\alpha(i)h_\alpha(m+n-i):)$$

$$+ \sum_{n<i<m+n}(i-n)(-i)(h_\alpha,h_\alpha)\delta_{m+n,0}.$$

2)

$$[L_{E_8}(m), \frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}\sum_{i\in\mathbb{Z}}:e_\alpha(i)f_\alpha(n-i):]$$

$$= \frac{1}{30}(\sum_{i<n}((-i)e_\alpha(i+m)f_\alpha(n-i)+(i-n)e_\alpha(i)f_\alpha(m+n-i))$$

$$+ \sum_{i>n}((i-n)f_\alpha(m+n-i)e_\alpha(i)-if_\alpha(n-i)e_\alpha(m+i))-ne_\alpha(m+n)f_\alpha(0))$$

$$= \frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}((-i):e_\alpha(i+m)f_\alpha(n-i):+(i-n):e_\alpha(i)f_\alpha(m+n-i):)$$

$$+ \sum_{i=n+1}^{m+n}(i-n)f_\alpha(m+n-i)e_\alpha(i) - \sum_{i=n}^{m+n-1}(i-n)e_\alpha(i)f_\alpha(m+n-i)$$

$$+ me_\alpha(m+n)f_\alpha(0))$$

$$= \frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}((-i):e_\alpha(i+m)f_\alpha(n-i):+(i-n):e_\alpha(i)f_\alpha(m+n-i):)$$

$$+ m([f_\alpha,e_\alpha](m+n)) + \sum_{n<i<m+n}(i-n)([f_\alpha,e_\alpha](m+n)$$

$$+ (m+n-i)(f_\alpha,e_\alpha)\delta_{m+n-i,-i})).$$

3) By the same way, we get

$$[L_{E_8}(m), \frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}\sum_{i\in\mathbb{Z}}:f_\alpha(i)e_\alpha(n-i):]$$

$$= \frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}((-i):f_\alpha(i+m)e_\alpha(n-i):+(i-n):f_\alpha(i)e_\alpha(m+n-i):)$$

$$+ m([e_\alpha,f_\alpha](m+n)) + \sum_{n<i<m+n}(i-n)([e_\alpha,f_\alpha](m+n)$$

$$+ (m+n-i)(e_\alpha,f_\alpha)\delta_{m+n-i,-i})).$$

Add to above 1), 2), 3), we have

If $m+n\neq 0$, there is

$$\begin{aligned}
&[L_{E_8}(m), L_{D_8}(n)]\\
&=\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\sum_{i\in\mathbb{Z}}(-i):h_\alpha(i+m)h_\alpha(n-i):+(i-n):h_\alpha(i)h_\alpha(m+n-i):)\\
&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}(-i):e_\alpha(i+m)f_\alpha(n-i):+(i-n):e_\alpha(i)f_\alpha(m+n-i):)\\
&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{i\in\mathbb{Z}}(-i):f_\alpha(i+m)e_\alpha(n-i):+(i-n):f_\alpha(i)e_\alpha(m+n-i):)\\
&=\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{l\in\mathbb{Z}}(m-l):h_\alpha(l)h_\alpha(n+m-l):+\sum_{i\in\mathbb{Z}}(i-n):h_\alpha(i)h_\alpha(m+n-i):)\\
&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{l\in\mathbb{Z}}(m-l):e_\alpha(l)f_\alpha(n+m-l):+\sum_{i\in\mathbb{Z}}(i-n):e_\alpha(i)f_\alpha(m+n-i):)\\
&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{l\in\mathbb{Z}}(m-l):f_\alpha(l)e_\alpha(n+m-l):+\sum_{i\in\mathbb{Z}}(i-n):f_\alpha(i)e_\alpha(m+n-i):)\\
&=\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}(\sum_{i\in\mathbb{Z}}((m-n):h_\alpha(i)h_\alpha(n+m-i):)\\
&+\frac{1}{30}\sum_{\alpha\in\Delta^+_{D_8}}((m-n)\sum_{i\in\mathbb{Z}}(:f_\alpha(i)e_\alpha(n+m-i):+:f_\alpha(i)e_\alpha(m+n-i):))\\
&=(m-n)L_{D_8}(m+n).
\end{aligned}$$

If $m+n=0$, there is

$$\begin{aligned}
[L_{E_8}(m),L_{D_8}(n)] &= -2nL_{D_8}(0)+\frac{1}{60}\sum_{\alpha\in\underline{\Delta}^+_{D_8}}\sum_{n<i<m+n}(i-n)(-i)(h_\alpha,h_\alpha)\\
&+\frac{1}{15}\sum_{\alpha\in\Delta^+_{D_8}}(\sum_{n<i<m+n}(i-n)(-i))\\
&=-2nL_{D_8}(0)+\frac{16}{60}\sum_{n<i<m+n}(i-n)(-i)+\frac{56}{15}\sum_{n<i<m+n}(i-n)(-i)\\
&=-2nL_{D_8}(0)+4\sum_{n<i<m+n}(i-n)(-i)\\
&=-2nL_{D_8}(0)-\frac{n^3-n}{12}c_{E_8}.
\end{aligned}$$

Therefore we get the relation

$$[L_{E_8}(m), L_{D_8}(n)] = (m-n)L_D(m+n) - \frac{n^3-n}{12}\delta_{m+n,0}c_{D_8}. \tag{5.20}$$

As the same way of the case of $m>0$, we know that if $m<0$, then $L_{E_8}(m), L_{D_8}(n)$ also satisfy the relation (5.20). Therefore, for $m \neq 0$, there is $[L_{E_8}(m), L_{D_8}(0)] = mL_{D_8}(m)$. And since $L_{D_8}(0) = L_{E_8}(0)$ on $L_{E_8}(1,0)$, we have $[L_{E_8}(m), L_{E_8}(0)] = [L_{E_8}(m), L_{D_8}(0)] = mL_{D_8}(m)$. As a conformal vector of $L_{E_8}(1,0)$, there is $[L_{E_8}(m), L_{E_8}(0)] = mL_{E_8}(m)$. So we have $L_{D_8}(n) = L_{E_8}(n)$, for $n \in \mathbb{Z}$, as operators of $L_{E_8}(1,0)$. Finally we have $Y(\omega_{E_8}, t) = Y(\omega_{D_8}, t)$ as vertex operators of $L_{E_8}(1,0)$.

Since $L_{E_8}(1,0)$ is a simple vertex operator algebra, we know that $\omega_{E_8} = \omega_{D_8}$ as conformal vectors of $L_{E_8}(1,0)$ by Lemma 5.5 and Proposition 5.12. Therefore we can get

Theorem 5.13. *$(L_{D_8}(1,0), Y, \mathbf{1}, \omega_{D_8})$ is a vertex operator subalgebra of vertex operator algebra $(L_{E_8}(1,0), Y, \mathbf{1}, \omega_{E_8})$.*

Moreover, we determine the decomposition of $L_{E_8}(1,0)$ into a direct sum of simple $L_{D_8}(1,0)$-modules.

Lemma 5.14. *The lowest conformal weight of $L_{D_8}(1,0)$ is 0, and that of $L_{D_8}(1,\overline{\omega}_1)$ is $\frac{1}{2}$; The lowest conformal weights of $L_{D_8}(1,\overline{\omega}_7)$ and $L_{D_8}(1,\overline{\omega}_8)$ are both 1.*

Lemma 5.15. *The vector $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1}$ is a singular vector for $\widehat{\mathbf{g}}_{D_8}$ in $L_{E_8}(1,0)$.*

Proof. It is sufficient to show that

$$\begin{aligned}
&e_i(0)\cdot e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1} = 0, \quad i = 1,2,\cdots,8,\\
&f_\theta(1)\cdot e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1} = 0.
\end{aligned}$$

Where $e_i := e_{\beta_i}$ for $\beta_i \in \Pi_{D_8}$ which is the simple root set of Lie algebra $\mathbf{g}_{D_8}$. And $\theta = \epsilon_1 + \epsilon_2$ is the highest root of $\mathbf{g}_{D_8}$.

It can easily be checked that

$$e_i(0)\cdot e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1} = [e_{\beta_i}(0), e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)]\cdot\mathbf{1} = 0,$$

Similarly, we can show that

$$\begin{aligned}
&f_\theta(1)\cdot e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1} = [f_{\epsilon_1+\epsilon_2}(1), e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)]\cdot\mathbf{1}\\
&= ([f_{\epsilon_1+\epsilon_2}, e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}](0) + (f_{\epsilon_1+\epsilon_2}(1), e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}))\cdot\mathbf{1}\\
&= 0,
\end{aligned}$$

hence $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1}$ is a singular vector for $\widehat{\mathbf{g}}_{D_8}$ in $L_{E_8}(1,0)$.

As a result, we have the following decomposition of $L_{E_8}(1,0)$.

Theorem 5.16. *As an $L_{D_8}(1,0)$-module, $L_{E_8}(1,0)$ can be decomposed into*

$$L_{E_8}(1,0) \cong L_{D_8}(1,0) \oplus L_{D_8}(1,\overline{\omega}_8). \tag{5.21}$$

Proof. From Theorem 5.13, it follows that $L_{E_8}(1,0)$ is an $L_{D_8}(1,0)$-module. Using Corollary 4.1 and the regularity of vertex operator algebra $L_{D_8}(1,0)$, we know $L_{E_8}(1,0)$ is a direct sum of copies of simple $L_{D_8}(1,0)$- modules $L_{D_8}(1,0), L_{D_8}(1,\overline{\omega}_1), L_{D_8}(1,\overline{\omega}_7), L_{D_8}(1,\overline{\omega}_8)$. By Lemma 5.15, we know $\mathbf{1}$ and $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1}$ are singular vectors for $\widehat{\mathbf{g}}_{D_8}$ in $L_{E_8}(1,0)$ which generate the following $L_{D_8}(1,0)$-modules:

$$U(\widehat{\mathbf{g}}_{D_8})\mathbf{1} \cong L_{D_8}(1,0);$$
$$U(\widehat{\mathbf{g}}_{D_8})e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1} \cong L_{D_8}(1,\overline{\omega}_8).$$

From Lemma 5.14, it follows that the lowest conformal weights of simple $L_{D_8}(1,0)$-modules (4.3) are $0, \frac{1}{2}, 1, 1$, respectively. So we know that $L_{E_8}(1,0)$ is a direct sum of copies of simple $L_{D_8}(1,0)$-modules $L_{D_8}(1,0)$, $L_{D_8}(1,\overline{\omega}_7), L_{D_8}(1,\overline{\omega}_8)$. As similar as the proof of Lemma 5.15, we can prove that $\mathbf{1}$ and $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_8)}(-1)\mathbf{1}$ are only singular vectors for $\widehat{\mathbf{g}}_{D_8}$ in $L_{E_8}(1,0)$, which implies

$$L_{E_8}(1,0) \cong L_{D_8}(1,0) \oplus L_{D_8}(1,\overline{\omega}_8).$$

Remark 5.17. 1) It follows that from above Theorem 5.16 the extension of vertex operator algebra $L_{D_8}(1,0)$ by $L_{D_8}(1,\overline{\omega}_8)$ is a vertex operator algebra, which is isomorphic to $L_{E_8}(1,0)$.

2) Theorem 5.16 also implies that $\widehat{\mathbf{g}}_{D_8}$-module $L_{E_8}(1,0)$, which is considered as a module for Lie subalgebra $\widehat{\mathbf{g}}_{D_8}$ of $\widehat{\mathbf{g}}_{E_8}$, decomposes into the finite direct sum of $\widehat{\mathbf{g}}_{D_8}$-modules.

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Yan-Jun Chu, Zhu-Jun Zheng
Department of Mathematics
South China University of Technology
Guangzhou 510641, P. R. China
and
Institute of Mathematics

Henan University
Kaifeng 475001, P. R. China
E-mail: zhengzj@scut.edu.cn